FP3 June 09 M.A Kerime 2

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1. Solve the equation

$$7\operatorname{sech} x - \tanh x = 5$$

Give your answers in the form $\ln a$ where a is a rational number.

(5)

$7\operatorname{sech} x - \tanh x = 5$

$\therefore \dfrac{7}{\cosh x} - \dfrac{\sinh x}{\cosh x} = 5$

$\therefore 7 - \sinh x = 5\cosh x$

$\therefore 7 = 5\cosh x + \sinh x$

$\therefore 7 = \dfrac{5e^x + 5e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2}$

$\therefore 14 = 6e^x + 4e^{-x}$

$\therefore 14e^x = 6e^{2x} + 4$

$\therefore 6e^{2x} - 14e^x + 4 = 0 \Rightarrow 3e^{2x} - 7e^x + 2 = 0$

$(e^x - 2)(3e^x - 1) = 0$

$e^x = 2 \Rightarrow x = \ln 2$

$e^x = \frac{1}{3} \Rightarrow x = \ln \frac{1}{3}$

2.

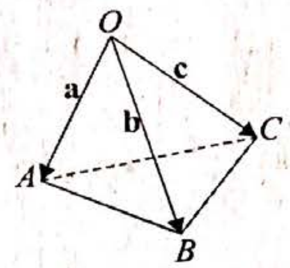

Figure 1

The points A, B and C have position vectors **a**, **b** and **c** respectively, relative to a fixed origin O, as shown in Figure 1.

It is given that

$$\mathbf{a} = \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{c} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Calculate

(a) $\mathbf{b} \times \mathbf{c}$, **(3)**

(b) $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$, **(2)**

(c) the area of triangle OBC, **(2)**

(d) the volume of the tetrahedron $OABC$. **(1)**

(a) $\underset{\sim}{b} \times \underset{\sim}{c} = \times \begin{matrix} 3 & -1 & 1 & 3 & -1 & 1 \\ 2 & 1 & -1 & 2 & 1 & -1 \end{matrix}$

$= \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$

(b) $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = 5$

Question 2 continued

(c) Area $\Delta OBC = \frac{1}{2}|\underset{\sim}{b} \times \underset{\sim}{c}| = \frac{1}{2}\sqrt{0^2+5^2+5^2}$

$= \frac{5}{2}\sqrt{2}$ units^2

(d) $Vol = \frac{1}{6}|\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})| = \cancel{\frac{1}{6}\sqrt{\ }}$ $\frac{1}{6} \times 5$

$= \frac{5}{6}$ units^3

blank

3.

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

(a) Show that 7 is an eigenvalue of the matrix **M** and find the other two eigenvalues of **M**.

(5)

(b) Find an eigenvector corresponding to the eigenvalue 7.

(4)

(a). $M - \cancel{\lambda} I = \begin{pmatrix} 6-\lambda & 1 & -1 \\ 0 & 7-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{pmatrix}$

$\det(M-\lambda I) = 0$

$\Rightarrow \det(M-\lambda I) = (6-\lambda)(7-\lambda)(2-\lambda) - 1(0)$

$+ -1(0 - 3(7-\lambda))$

$= (6-\lambda)(7-\lambda)(2-\lambda) - (-21+3\lambda)$

$= (6-\lambda)(7-\lambda)(2-\lambda) + 3(7-\lambda)$

$= (7-\lambda)((6-\lambda)(2-\lambda) + 3)$

$= (7-\lambda)(\lambda^2 - 8\lambda + 15) = 0$

$\Rightarrow 7-\lambda = 0 \therefore \lambda = 7$ is an eigenvalue

Also $\lambda^2 - 8\lambda + 15 = 0$

$\Rightarrow (\lambda-5)(\lambda-3) = 0 \quad \therefore \lambda = 5, \lambda = 3$ are eigenvalues blank

Question 3 continued

(b) $Mx = 7x$

$$\Rightarrow \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x \\ 7y \\ 7z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 6x+y-z \\ 7y \\ 3x-y+2z \end{pmatrix} = \begin{pmatrix} 7x \\ 7y \\ 7z \end{pmatrix}$$

$\therefore \quad 6x+y-z = 7x \Rightarrow x = y-z$

$7y = 7y$

$3x-y+2z = 7z \Rightarrow 3x-y = 5z$

$\Rightarrow z = \frac{3}{5}x - \frac{y}{5}$

$\therefore x = y - \frac{3}{5}x + \frac{y}{5} \Rightarrow \frac{8}{5}x = \frac{6}{5}y$

$\Rightarrow x = \frac{3}{4}y$

$z = y - x = \frac{4}{3}x - x = \frac{1}{3}x \Rightarrow z = \frac{1}{3}x$

$x = 1 \Rightarrow y = \frac{4}{3} \quad z = \frac{1}{3}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, Let $x = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ be the eigenvector

Leave blank

4. Given that $y = \text{arsinh}(\sqrt{x}), \quad x > 0,$

(a) find $\frac{dy}{dx}$, giving your answer as a simplified fraction. **(3)**

(b) Hence, or otherwise, find

$$\int_{\frac{1}{4}}^{4} \frac{1}{\sqrt{[x(x+1)]}} dx,$$

giving your answer in the form $\ln\left(\frac{a+b\sqrt{5}}{2}\right)$, where a and b are integers. **(6)**

4 (a). $y = \text{arsinh}(\sqrt{x})$

$c^2 - s^2 = 1$

$c^2 = 1$

$\therefore \sinh y = \sqrt{x}$

$\cancel{c^2 + s^2 = 1}$

$\therefore \frac{\partial y}{\partial x} \cosh y = \frac{1}{2} x^{-\frac{1}{2}}$

$\cancel{c^2 = 1 - s^2}$

$\therefore \frac{\partial y}{\partial x} = \frac{1}{2\sqrt{x} \cosh y}$

$\therefore \frac{\partial y}{\partial x} = \frac{1}{2\sqrt{x}\sqrt{1 + \sinh^2 y}} = \frac{1}{2\sqrt{x}\sqrt{1+x}}$

$\therefore \frac{\partial y}{\partial x} = \frac{1}{2\sqrt{x+x^2}}$

(b) $\int_{1/4}^{4} \frac{1}{\sqrt{x(x+1)}} \partial x = 2 \int_{1/4}^{4} \frac{1}{2\sqrt{x+x^2}} \partial x$

$= 2 \int_{1/4}^{4} \frac{\partial}{\partial x} (\text{arsinh}\sqrt{x}) \partial x$

$$= 2\left[\operatorname{arsinh}\sqrt{x}\right]_{\frac{1}{4}}^{4}$$

$$= 2\left(\operatorname{arsinh} 2 - \operatorname{arsinh}\tfrac{1}{2}\right)$$

$$= 2\left(\ln\left(\sqrt{5} + 2\right) - \ln\left(\tfrac{1}{2} + \sqrt{\tfrac{5}{4}}\right)\right)$$

$$= 2\left(\ln\left(2+\sqrt{5}\right) - \ln\left(\frac{1+\sqrt{5}}{2}\right)\right)$$

$$= 2\ln\left(\frac{3+\sqrt{5}}{2}\right) = \ln\left[\left(\frac{3+\sqrt{5}}{2}\right)^{2}\right]$$

$$= \ln\left(\frac{7+3\sqrt{5}}{2}\right)$$

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5.

$$I_n = \int_0^5 \frac{x^n}{\sqrt{(25-x^2)}}\,dx\,, \qquad n \geqslant 0$$

(a) Find an expression for $\int \frac{x}{\sqrt{(25-x^2)}}\,dx$, $0 \leqslant x \leqslant 5$.

(2)

(b) Using your answer to part (a), or otherwise, show that

$$I_n = \frac{25(n-1)}{n} I_{n-2} \qquad n \geqslant 2$$

(5)

(c) Find I_4 in the form $k\pi$, where k is a fraction.

(4)

5(a). $\int \frac{x}{\sqrt{25-x^2}}\,\partial x = \int x\,(25-x^2)^{-1/2}$

$= -\frac{1}{2}\int -2x\,(25-x^2)^{-1/2}\,\partial x$

$= -\frac{1}{2}\left(\frac{(25-x^2)^{1/2}}{1/2}\right) + C$

$= -\sqrt{25-x^2} + C$

(b) $I_n = \int_0^5 \frac{x^n}{\sqrt{25-x^2}}\,\partial x = \int_0^5 \frac{x^{n-1}\cdot x}{\sqrt{25-x^2}}\,\partial x$

Let $u = x^{n-1}$ $\quad u' = (n-1)x^{n-2}$

$v' = \frac{x}{\sqrt{25-x^2}}$ $\quad v = -\sqrt{25-x^2}$

$\therefore I_n = \left[-x^{n-1}\sqrt{(25-x^2)}\right]_0^5 + (n-1)\int_0^5 x^{n-2}\sqrt{25-x^2}\,dx$

$\therefore I_n = (n-1)\int_0^5 x^{n-2}(25-x^2)^{-\frac{1}{2}}(25-x^2)\,dx$

$\therefore I_n = (n-1)\int_0^5 25x^{n-2}(25-x^2)^{-1/2} - x^n(25-x^2)^{-\frac{1}{2}}dx$

$\therefore I_n = (n-1)\left(25I_{n-2} - I_n\right)$

$\therefore I_n = 25(n-1)I_{n-2} - (n-1)I_n$

$\therefore nI_n = 25(n-1)I_{n-2}$

$\Rightarrow I_n = \frac{25(n-1)}{n}I_{n-2}$ // as required.

(c) $I_4 = \frac{75}{4}I_2 = \frac{75}{4}\left(\frac{25}{2}I_0\right)$

$\therefore I_4 = \frac{1875}{8}\int_0^5 \frac{1}{\sqrt{25-x^2}}\,dx = \frac{1875}{8}\left[\arcsin\frac{x}{5}\right]_0^5$

$= \frac{1875}{8}(\arcsin 1)$

$= \frac{1875}{16}\pi$ //

Q5

(Total 11 marks)

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6. The hyperbola H has equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where a and b are constants.

The line L has equation $y=mx+c$, where m and c are constants.

(a) Given that L and H meet, show that the x-coordinates of the points of intersection are the roots of the equation

$$(a^2m^2-b^2)x^2+2a^2mcx+a^2(c^2+b^2)=0$$

(2)

Hence, given that L is a tangent to H,

(b) show that $a^2m^2=b^2+c^2$.

(2)

The hyperbola H' has equation $\frac{x^2}{25}-\frac{y^2}{16}=1$.

(c) Find the equations of the tangents to H' which pass through the point (1, 4).

(7)

6 (a). $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ & $y=mx+c$ @ intersection.

~~&~~

$\therefore \frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$

$\therefore \frac{x^2}{a^2}-\frac{m^2x^2+2mcx+c^2}{b^2}=1$

$\therefore \frac{x^2b^2}{a^2b^2}-\frac{a^2m^2x^2+2a^2mcx+a^2c^2}{a^2b^2}=1$

$\therefore x^2b^2-a^2m^2x^2-2a^2mcx-a^2c^2=a^2b^2$

$\Rightarrow a^2m^2x^2-x^2b^2+2a^2mcx+a^2c^2+a^2b^2=0$

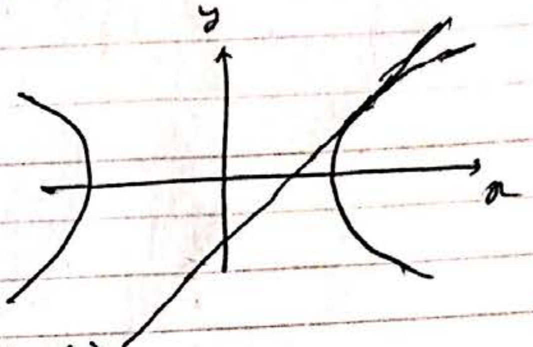
$\therefore (a^2m^2-b^2)x^2+2a^2mcx+a^2(c^2+b^2)=0$

as required.

Question 6 continued

(b)

L and H intersect once.



$\therefore$ Quadratic eqn in part (a) has only one solution. $\Rightarrow$ discriminant $=0$

$\therefore (2a^2mc)^2 - 4(a^2m^2-b^2)(a^2(c^2+b^2)) = 0$

$\therefore 4a^4m^2c^2 - 4a^2(a^2m^2-b^2)(c^2+b^2) = 0$

$\therefore 4a^4m^2c^2 - 4a^2(a^2m^2c^2 + a^2m^2b^2 - b^2c^2 - b^4) = 0$

$\therefore 4a^4m^2c^2 - 4a^4m^2c^2 - 4a^4m^2b^2 + 4a^2b^2c^2 + 4a^2b^4 = 0$

$\therefore -4a^4m^2b^2 + 4a^2b^2c^2 + 4a^2b^4 = 0$

$\therefore 4a^2b^2(c^2 - a^2m^2 + b^2) = 0$

$\therefore c^2 - a^2m^2 + b^2 = 0 \Rightarrow c^2 + b^2 = a^2m^2$

as required.

(c) $\frac{x^2}{25} - \frac{y^2}{16} = 1 \Rightarrow a^2 = 25,\ b^2 = 16$

$a^2m^2 = b^2 + c^2$

$\therefore 25m^2 = 16 + c^2$

& $a^2m^2 - b^2 = 25m^2 - 16$

$(1,4) \Rightarrow 4 = m + c$

$\Rightarrow c = 4 - m$

$\therefore 25m^2 = 16 + (4-m)^2$

$\therefore \quad 25m^2 = m^2 - 8m + 32$

$\therefore \quad \cancel{m^2 - 8m}$

$24m^2 + 8m - 32 = 0$

$\therefore \cancel{6} \quad 3m^2 + m - 4 = 0$

$\cancel{(3m \quad)(m \quad)}$

$\cancel{m} \quad (m-1)(3m+4) = 0$

$\therefore \quad m = 1 \quad \& \quad m = -\frac{4}{3}$

$\therefore \quad c = 4 - 1 = 3$

$\& \quad c = 4 + \frac{4}{3} = \frac{16}{3}$

$\therefore \quad y = x + 3$

$\& \quad y = -\frac{4}{3}x + \frac{16}{3}$

7. The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} \alpha \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

If the lines l_1 and l_2 intersect, find

(a) the value of α, **(4)**

(b) an equation for the plane containing the lines l_1 and l_2, giving your answer in the form $ax + by + cz + d = 0$, where a, b, c and d are constants. **(4)**

For other values of α, the lines l_1 and l_2 do not intersect and are skew lines.

Given that $\alpha = 2$,

(c) find the shortest distance between the lines l_1 and l_2. **(3)**

7. (a). $\underset{\sim}{r} = \begin{pmatrix} 1-\lambda \\ 3\lambda-1 \\ 2+4\lambda \end{pmatrix}$ & $\underset{\sim}{r} = \begin{pmatrix} \alpha \\ -4+3\mu \\ 2\mu \end{pmatrix}$

$3\lambda - 1 = -4 + 3\mu$

& $2 + 4\lambda = 2\mu \Rightarrow 3 + 6\lambda = 3\mu$

$\therefore\ 3\lambda - 1 = -4 + 3 + 6\lambda$

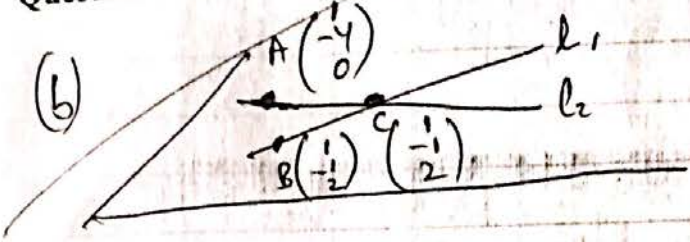
$\Rightarrow 3\lambda = 0$

$\Rightarrow \lambda = 0 \quad \mu = 1$

$\therefore\ 1 - \lambda = \alpha \Rightarrow \alpha = 1$

(b)

(b)



Let C be intersection

Intersection @ C

$\begin{pmatrix}1\\-1\\2\end{pmatrix}$

Let A be point on l_1 : $\begin{pmatrix}1\\-4\\0\end{pmatrix}$

Let B be point on l_2 : $\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

$\vec{AC} = \begin{pmatrix}1\\-1\\2\end{pmatrix} - \begin{pmatrix}1\\-4\\0\end{pmatrix} = \begin{pmatrix}0\\3\\2\end{pmatrix}$

$\vec{BC} = \begin{pmatrix}1\\-1\\2\end{pmatrix} - ($

(b). $n = \begin{pmatrix}-1\\3\\4\end{pmatrix} \times \begin{pmatrix}0\\3\\2\end{pmatrix} =$

$$= \begin{pmatrix}-6\\2\\-3\end{pmatrix}$$

$\therefore r \cdot n = \cancel{-6}\ \ r \cdot \begin{pmatrix}-6\\2\\-3\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix} \cdot \begin{pmatrix}-6\\2\\-3\end{pmatrix}$

$\therefore -6x + 2y - 3z = -6 - 2 - 6$

$\therefore -6x + 2y - 3z = -14$

$\therefore 6x - 2y + 3z - 14 = 0$

Question 7 continued

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use:

(c) $d = \left| \dfrac{(\underline{a}-\underline{c})\cdot(\underline{b}\times\underline{d})}{|\underline{b}\times\underline{d}|} \right|$

$$d = \left| \frac{\left[\begin{pmatrix}1\\-1\\2\end{pmatrix} - \begin{pmatrix}2\\-4\\0\end{pmatrix}\right] \cdot \left[\begin{pmatrix}-6\\2\\-3\end{pmatrix}\right]}{\sqrt{6^2+2^2+3^2}} \right|$$

$$\therefore d = \frac{\begin{pmatrix}-1\\3\\2\end{pmatrix} \cdot \begin{pmatrix}-6\\2\\-3\end{pmatrix}}{\sqrt{6^2+2^2+3^2}} = \frac{6}{7}$$

$\therefore$ distance $= \frac{6}{7}$

8. A curve, which is part of an ellipse, has parametric equations

$$x = 3\cos\theta, \quad y = 5\sin\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

The curve is rotated through 2π radians about the x-axis.

(a) Show that the area of the surface generated is given by the integral

$$k\pi\int_0^{\alpha} \sqrt{(16c^2 + 9)}\,\mathrm{d}c, \quad \text{where } c = \cos\theta,$$

and where k and α are constants to be found.

(6)

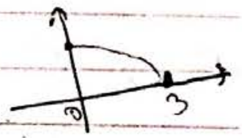
(b) Using the substitution $c = \frac{3}{4}\sinh u$, or otherwise, evaluate the integral, showing all of your working and giving the final answer to 3 significant figures.

(5)

(a) $x = 3\cos\theta \therefore \frac{\partial x}{\partial \theta} = -3\sin\theta \therefore \left(\frac{\partial x}{\partial\theta}\right)^2 = 9\sin^2\theta$

$y = 5\sin\theta \therefore \frac{\partial y}{\partial\theta} = 5\cos\theta \therefore \left(\frac{\partial y}{\partial\theta}\right)^2 = 25\cos^2\theta$

$S = -2\pi\int_{\pi/2}^{0} y\sqrt{\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2}\,\partial\theta$



Limits from $\frac{\pi}{2}$ to 0 gives $-$ve S $\therefore$ add a '$-$' sign in front of integral

Let $s = \sin\theta$ Let $c = \cos\theta$

$\therefore S = -2\pi\int_{\pi/2}^{0} 5s\sqrt{9s^2 + 25c^2}\,\partial\theta$

$= -10\pi\int_{\pi/2}^{0} s\sqrt{9 - 9c^2 + 25c^2}\,\partial\theta$

$S = -10\pi\int_{\pi/2}^{0} s\sqrt{9 + 16c^2}\,\partial\theta$

Leave blank

Question 5 continued

Now let c represent a variable:

~~= 10π~~ $c = \cos\theta$

$$\frac{\partial c}{\partial \theta} = -\sin\theta$$

$$\therefore \partial\theta = -\frac{1}{\sin\theta} dc = -\frac{1}{s} dc$$

$\therefore$ ~~$s = 10\pi \int_{\pi}^{0}$~~ Consider limits:

~~$c = 0 \Rightarrow \cos\theta = 0 \Rightarrow$~~

$$c = \cos(0) = 1$$
$$c = \cos(\pi/2) = 0 \quad \Rightarrow s = \int_0^1$$

$$\therefore S = -10\pi \int_{\pi/2}^{0} s\sqrt{9+16c^2}\, d\theta = -10\pi \int_0^1 s\sqrt{9+16c^2} \cdot \frac{-1}{s} dc$$

$$\therefore S = -10\pi \int_0^1 -\sqrt{16c^2+9}\, dc$$

$$\therefore S = +10\pi \int_0^1 \sqrt{16c^2+9}\, dc \qquad \alpha = 1$$

Q5

(Total 10 marks)

(b) $10\pi \int_0^1 \sqrt{16\left(\frac{3}{4}\sinh u\right)^2 + 9}\ dc$

$c = \frac{3}{4}\sinh u$

$\therefore \frac{dc}{du} = \frac{3}{4}\cosh u$

$\therefore dc = \frac{3}{4}\cosh u\, du$

$\therefore S = 10\pi \int \sqrt{9\sinh^2 u + 9} \cdot \frac{3}{4}\cosh u\, du$

$= 10\pi \int \frac{9}{4}\sqrt{\sinh^2 u + 1}\ \cosh u\, du$

$\cancel{c^2 - s^2 =}$

$c^2 + s^2 = \cosh 2x$

$c^2 + c^2 - 1$

$2c^2 - 1$

$= 10\pi \int \frac{9}{4}\cosh^2 u\, du$

$= \frac{90}{8}\pi \int \cosh 2u + 1\, du$

$= \frac{90}{8}\pi \left[\frac{1}{2}\sinh 2u + u\right]$

Consider limits: $\cancel{c = \frac{3}{4}\sinh(1) = \frac{3}{4}\frac{1}{2}(e^1 - e^{-1})}$

$\cancel{= \frac{3}{8}\left(e - \frac{1}{e}\right)}$

$\cancel{c = \frac{3}{4}\sinh}$ $\quad 1 = \frac{3}{4}\sinh u \Rightarrow u = \operatorname{arsinh}\frac{4}{3} = \ln 3$

$0 = \frac{3}{4}\sinh u \Rightarrow u = \operatorname{arsinh}(0) = 0$

$$\therefore \frac{90}{8}\pi \left[\frac{1}{2}\sinh 2u + u \right]_{\operatorname{arsinh}(0) = \ln(1) = 0}^{\operatorname{arsinh}\frac{4}{3} = \cancel{\ln\left(\frac{4}{3} + \sqrt{}\right)} = \ln 3}$$

$$= \frac{90}{8}\pi \left(\left[\frac{1}{2}\sinh 2u + u \right]_0^{\ln 3} \right.$$

$$= \frac{90}{8}\pi \left(\frac{1}{2}\sinh \ln 9 + \ln 3 \right)$$

$$= \frac{90}{8}\pi \left[\frac{1}{2} \cdot \frac{1}{2}\left(e^{\ln 9} - e^{-\ln 9}\right) + \ln 3 \right]$$

$$= \frac{90}{8}\pi \left[\frac{1}{4}\left(9 - \frac{1}{9}\right) + \ln 3 \right]$$

$$= 117.36\ldots = 117 \quad (3sf)$$